ON THE COMPLEXITY OF FOUR POLYHEDRAL

SET CONTAINMENT PROBLEMS

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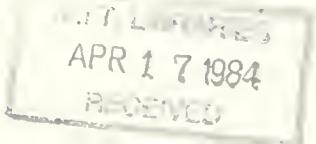Sloan W.P. No. 1527-84   January 1984

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Abstract

A nonempty closed convex polyhedron X can be represented either as
$X = \{x:Ax \leq b\}$, where (A,b) are given, in which case X is called an H-cell, or in
the form $X = \{x:x=U\lambda+V\mu, \Sigma\lambda_j=1, \lambda \geq 0, \mu \geq 0\}$, where (U,V) are given, in which case X
is called a W-cell. This note discusses the computational complexity of certain
set containment problems. The problems of determining if $X \subseteq Y$, where (i) X is
an H-cell and Y is a closed solid ball, (ii) X is an H-cell and Y is a W-cell,
or (iii) X is a closed solid ball and Y is a W-cell, are all shown to be
NP-complete, essentially verifying a conjecture of Eaves and Freund.
Furthermore, the problem of determining whether an integer point lies in a
W-cell is shown to be NP-complete, demonstrating that regardless of the repre-
sentation of X as an H-cell or W-cell, this integer containment problem is
NP-complete.

1. Introduction and Preliminaries

A nonempty closed convex polyhedron X can be represented either in the form $X = \{x : Ax \leq b\}$, where (A,b) are given, in which case X is called an <u>H-cell</u> (H for halfspaces), or in the form $X = \{x : x = U\lambda + V\mu, \Sigma\lambda_j = 1, \lambda \geq 0, \mu \geq 0\}$ where (U,V) are given, in which case X is called a <u>W-cell</u> (W for weighting of points). The computational complexity of many problems related to polyhedra depend on the polyhedral representation as an H-cell or a W-cell. For example, consider a linear program, which can be stated as maximize $c^t x$ subject $x \in X$, where X is a polyhedron. If X is an H-cell, this is the usual linear program, whose solution time, while polynomial, is by no means negligible. However, if X is represented as a W-cell, the linear programming problem becomes trivial. As another example, consider the problem of testing if $\bar{x} \in X$ for a given $\bar{x}$, where X is a polyhedron. If X is an H-cell, the problem is trivial, whereas if X is a W-cell, the problem reduces to solving a linear program.

This note discusses the complexity of two types of problems. The first problem is the <u>set containment problem</u> (SCP), that of determining if $X \subseteq Y$, where X (resp. Y) is a <u>cell</u>, defined to be either a polyhedron (an H-cell or a W-cell), or a closed solid ball of the form $\{x : (x-c)^t(x-c) \leq r^2\}$, in which case X (resp. Y) is called a <u>B-cell</u>. There are nine forms of SCP corresponding to X and Y each being given as an H-cell, W-cell, or B-cell. For notational convenience, a particular form of SCP will be denoted, e.g., by (W,B), where X is a W-cell and Y is a B-cell. In Eaves and Freund [1], SCP is shown to be solvable as a linear program for the six forms (HH), (WH), (BH), (WW), (WB), and (BB), thus showing that these problems are solvable in polynomial time. Eaves and Freund also conjectured that the forms (HW), (BW), and (HB) are "intractable." In

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Section 2 of this article, we show that these three forms of SCP are NP-complete, thus essentially confirming the conjecture.

Section 3 addresses the computational complexity of the integer containment problem (ICP), that of finding an integer point in a given polyhedron X in the case that X is a W-cell. Karp [3] showed that when X is an H-cell, the corresponding ICP is NP-complete. Herein, it is shown that ICP is NP-complete when X is a W-cell.

The notation used is standard. Let R^n be n-dimensional Euclidean space. The Euclidean norm of $x \epsilon R^n$ is represented by $\| x \|$. Let $e = (1,1,1,...1)$ where the dimension is clear from the context. Let Q^{mxn}, Q^n be the set of rational mxn matrices and n-vectors, respectively. Define $\{a,b\}^n = \{x \epsilon R^n : x_j = a \text{ or } b, j=1,...n\}$.

2.   Three NP-complete Cases of the Set Containment Problem.

Consider the following version of the integer containment problem:

(ICP1)   Given: $\bar{A} \in Q^{mxn}$

Question: Is there a $\pi \epsilon \{-1,1\}^n$ that satisfies $\bar{A}\pi \leq e$?

This classical integer linear inequalities problem is NP-complete, even if m is restricted to be 2, as there is an elementary transformation from the number partition problem. The three set containment problems of interest, forms (HB), (HW), and (BW), can be stated formally as:

(HB)   Given: $(A,b,c,r^2) \epsilon (Q^{mxn}, Q^m, Q^n, Q^1)$

Question: Is $X \nsubseteq Y$, where $X = \{x \epsilon R^n : Ax \leq b\}$ and $Y = \{x \epsilon R^n : (x-c)^t (x-c) \leq r^2\}$?

(HW)   Given: $(A,b,U,V) \epsilon (Q^{mxn}, Q^m, Q^{nxk}, Q^{nxp})$

Question: Is $X \nsubseteq Y$, where $X = \{x \epsilon R^n : Ax \leq b\}$ and $Y = \{x \epsilon R^n : x = U\lambda + V\mu, e^t\lambda = 1, \lambda, \mu \geq 0\}$?

(BW)   Given: $(c, r^2, U, V) \epsilon (Q^n, Q^1, Q^{nxk}, Q^{nxp})$

Question: Is $X \nsubseteq Y$, where $X = \{x \epsilon R^n : (x-c)^t (x-c) \leq r^2\}$ and

$$Y = \{x \epsilon R^n : x = U\lambda + V\mu, e^t\lambda = 1, \lambda, \mu \geq 0\}?$$

Note that problems (HB), (HW), and (BW) are all elements of NP. To see this for the problem (HB), note that if $X \not\subseteq Y$, then there exists either an extreme ray $\bar{v}$ of X, or else there exists an extreme point $\bar{x}$ of X that is not an element of B. If the former is true, then there is a submatrix of M of A consisting of rows of A, and an index j such that $\bar{v}$ is the unique solution to $M\bar{v}=0$, $\bar{v}_j \neq 0$. The size required to record $\bar{v}$ then is polynomially bounded in the size of A. If the latter is true, $\bar{x}=M^{-1}d$ for some submatrix M of A and subvector d of e, and the size of $\bar{x}$ is polynomially bounded in the size of A. The test that $(\bar{x}-c)^t(\bar{x}-c) > r^2$ is obviously polynomially bounded in the size of A as well.

To see that problem (HW) is an element of NP, first note that if $X \not\subseteq Y$ there exists $\bar{x}$ or $\bar{v}$ as above. The test that $\bar{v} \notin \{y : y=U\lambda, \lambda \geq 0\}$ or the test that $\bar{x} \notin Y$ is equivalent to solving a linear program, which is polynomially bounded in the data $(\bar{v}, U)$ or $(\bar{x}, U, V)$.

Finally, if for a given instance of (BW), suppose that $X \not\subseteq Y$. Then, either Y has no interior in $\mathbb{R}^n$, or there is an $(n-1)$-facet of x such that the shortest Euclidean distance from the hyperplane containing this facet to c is larger than r^2. If the former is true, then the system $\pi U = \alpha e$, $\pi V = 0$ has a solution $\pi \neq 0$. This test is polynomial in the data (U,V), and the size of π will be polynomial in this data. If the latter is true, there exists a submatrix U' consisting of columns of U, and a submatrix V' consisting of columns of V, such that the hyperplane containing the facet in question is determined by a solution $(\bar{\pi}, \bar{\alpha})$ to $\pi U' = \alpha e$, $\pi V' = 0$, $\pi \neq 0$, where the hyperplane in question is $Z = \{x \mid \bar{\pi} \cdot x = \bar{\alpha}\}$. The size of the unique solution $(\bar{\pi}, \bar{\alpha})$ to the above system is polynomially bounded in the data (U,V), and the shortest Euclidean distance from c to Z is given by $\sqrt{\dfrac{|\bar{\alpha} - \bar{\pi} \cdot c|}{\bar{\pi} \cdot \bar{\pi}}}$. The test that $\dfrac{|\bar{\alpha} - \bar{\pi} \cdot c|}{\bar{\pi} \cdot \bar{\pi}} > r^2$ is also polynomially bounded in the data $(\bar{\alpha}, \bar{\pi}, c, r^2)$ and so problem (BW) is in the class NP.

Also note that if $X \not\subseteq Y$, then it follows from the convexity of Y and a separating hyperplane theorem that there exists $\bar{x} \in X$, and π, α such that (i) $\pi^t y \leq \alpha$ for all $y \in Y$, and (ii) $\pi^t \bar{x} > \alpha$.

Our main result in this section is the following:

THEOREM 1. The set containment problems (HB), (HW), and (BW) are NP-complete.

Before proceeding to the proofs, we define a few more terms and we state an elementary property concerning linear programs defined over the rationals.

For each matrix $\bar{A}$, let $P(\bar{A}) = \{x : \bar{A}x \leq e, -e \leq x \leq e\}$. Thus the integer containment problem ICP1 can be stated as follows: Does $P(\bar{A}) \cap \{-1,1\}^n = \phi$?

For a given rational matrix A, we will let $\max(A)$ denote the maximum absolute value of a numerator or denominator of a component of A: e.g., $\max(2/11, -14/2)) = 14$. (The numerator and divisor can have a common divisor.)

For two sets S, T, let $d(S,T)$ be the infinium of the distance between the two sets, where the supremum norm is used. In the proofs, we will use the following elementary lemma.

LEMMA 1. If $P(\bar{A}) \cap \{-1,1\}^n = \phi$, then $d(P(\bar{A}), \{-1,1\}^n) > (2\max(\bar{A})^{(n^3+n)} n!)^{-1}$.

PROOF. Let $z^* = d(P(\bar{A}), \{-1,1\}^n)$, and

$z^*(y) = d(P(\bar{A}), \{y\})$; then

$z^* = \min(z^*(y) : y \in \{-1,1\}^n)$, and

$z^*(y) = $ minimum z,
 subject to $z + (x_j - y_j) \geq 0$ $j = 1, \ldots, n$
 $z - (x_j - y_j) \geq 0$ $j = 1, \ldots, n$
 $x \in P(\bar{A})$.

We now claim that $z^*(y) > (2\max(\bar{A})^{(n^3+n)}(n!))^{-1}$ for any $y \in \{-1,1\}^n$. To see this, let

x^* be a point in $P(\bar{A})$ of minimum distance to y, and without loss of generality

we may assume that x^* is an extreme point of the feasible region of the above

linear program. Therefore $x^* = B^{-1}f$ where B is a row basis of the linear

program and f is a vector of 0's and 1's of the right-hand side components

corresponding to B. B can be written as $B = d^{-1}C$ where d is a common

denominator of B and C is an integral matrix. Because $B^{-1} = dC^{-1} = d\left(\dfrac{adj(C)}{det(C)}\right)$,

a denominator for B^{-1} is $det(C)$. Because $d \leq \max(\bar{A})^{n^2}$ and

$\max(C) \leq (\max(\bar{A})^{n^2+1})$, we obtain

$$det(C) \leq \max(C)^n n! \leq (\max(\bar{A})^{n^2+1})^n n! = \max(\bar{A})^{(n^3+n)} n! < 2\max(\bar{A})^{(n^3+n)} n! \ .$$

Thus this bound on $det(C)$, which is a denominator for any component of x^*,

provides a bound on $d(P(\bar{A}),\{-1,1\}^n)$. $\boxtimes$

Henceforth, for each $A \in Q^{mxn}$, let $M(A) = (2\max(\bar{A})^{(n^3+n)} n!)$. Note that

the size of $M(A)$ is $O(n^3 \log(1+\max(A)))$, which is polynomial in the size of A.

PROOF THAT (HB) IS NP-COMPLETE. Let $\bar{A}$ be an instance of ICP1, and

let $\varepsilon = [M(\bar{A})]^{-1}$. Let $X = P(\bar{A})$ and let $Y = \{y \in R^n : y^t y \leq n-\varepsilon\}$. Consider the

instance of (HB) of determining if $X \subseteq Y$.

Suppose first that $X \subseteq Y$. Then $\|x\|^2 \leq n-\varepsilon < n$ for any $x \in P(\bar{A})$ and thus

$P(\bar{A}) \cap \{-1,1\}^n = \phi$.

Conversely, suppose that $X \not\subseteq Y$. Let $x \in P(A)$ be selected so that $x \notin Y$.

Since $-e \leq x \leq e$ and $x^t x \geq n-\varepsilon$, it follows that $|x_j| \geq 1-\varepsilon$ for each j and thus

$d(x,\{-1,1\}^n) \leq \varepsilon$. It follows that $d(P(\bar{A}),\{-1,1\}^n) \leq \varepsilon$, and thus by Lemma 1, we

conclude that $P(\bar{A}) \cap \{-1,1\}^n \neq \phi$. $\boxtimes$

PROOF THAT (H,W) IS NP-COMPLETE. Let $\bar{A}$ be an instance of ICP1, and let $\varepsilon = [M(\bar{A})]^{-1}$. Let $X=P(\bar{A})$, and let $Y=\{y:\sum_{j=1}^{n}| y_j| \leq n-\varepsilon\}$. Note that Y may be polynomially represented as the W-cell $\{y:y=U\lambda,\lambda\geq 0,e^{t}\lambda=1\}$ by letting $U=[(n-\varepsilon)I,(n-\varepsilon)(-I)]$. Now consider the instance (H,W) of determining if $X\underline{c}Y$.

Suppose first that $X\underline{c}Y$. Then any $x\epsilon .P(\bar{A})$ must satisfy $\sum_j| x_j| \leq n-\varepsilon$ and thus $P(\bar{A})\cap\{-1,1\}^n = \phi$.

Suppose next that $X\not\subseteq Y$. Let $x\epsilon Y$ be chosen so that $\sum_j|x_j|>n-\varepsilon$. Since $-e\leq x\leq e$, it follows that $1-\varepsilon\leq| x_j|\leq 1$ for each $j=1,\ldots,n$ and thus $d(x,\{-1,1\}^n)\leq\varepsilon$. Therefore $d(P(\bar{A}),\{-1,1\}^n)\leq\varepsilon$, and thus by Lemma 1 we conclude that $P(\bar{A})\cap\{-1,1\}^n \neq \phi$. $\boxtimes$

PROOF THAT (B,W) IS NP-COMPLETE. Let $\bar{A}$ be an instance of ICP1 and let $\varepsilon = M(\bar{A})^{-1}$ Let $X=\{x\epsilon R^n: \|x\|^2\leq 1/(n-\varepsilon)\}$ and let $Y=\{y\epsilon R^n:\pi^{t}y\leq 1 \text{ for all } \pi\epsilon P(\bar{A})\}$. Y can be represented as the W-cell $Y'=\{y:y=\lambda^1-\lambda^2+\bar{A}^{t}\lambda^3,\lambda^1,\lambda^2,\lambda^3\geq 0,e^{t}\lambda^1+e^{t}\lambda^2+e^{t}\lambda^3=1\}$. (It is easy to see that $Y'\underline{c}Y$ by premultiplying any $y\epsilon Y'$ by $\pi\epsilon P(\bar{A})$. To show that $Y\underline{c}Y'$ one can assume that $y'\not\in Y'$ and use linear programming duality to show that $y'\not\in Y$.) Consider the instance of (BW) of determining if $X\underline{c}Y$.

Suppose first that $P(\bar{A})\cap\{-1,1\}^n\neq \phi$, and let $v\epsilon P(\bar{A})\cap\{-1,1\}^n$. Let $\bar{v}=(n-\varepsilon)^{-1/2}\|v\|^{-1}v$. Note first that $\bar{v}^{t}\bar{v}=(n-\varepsilon)^{-1}$ and so $\bar{v}\epsilon X$. Also note that $v^{t}\bar{v}=(n-\varepsilon)^{-1/2}\|v\|>1$, and so $\bar{v}\not\in Y$. We conclude in this case that $X\not\subseteq Y$. Thus if $X\underline{c}Y$, $P(\bar{A})\cap\{-1,1\}^n = \phi$.

Next consider the case that $X\not\subseteq Y$. In this case there exists $x\epsilon\bar{X}$ and $\pi\epsilon P(\bar{A})$ such that $\pi^{t}x>1$. Moreover the value of $\bar{x}\epsilon X$ which maximizes $\pi^{t}\bar{x}$ is uniquely given by $\bar{x}=(n-\varepsilon)^{-1/2}\|\pi\|^{-1}\pi$ whenever $\pi\neq 0$. Thus we may assume without loss of generality that $x=(n-\varepsilon)^{-1/2}\|\pi\|^{-1}\pi$. It follows that

$\|\pi\|^2 = \pi^t\pi = (n-\varepsilon)^{\frac{1}{2}}\|\pi\|$ $\pi^t x > (n-\varepsilon)^{\frac{1}{2}}\|\pi\|$, and thus $\|\pi\|^2 > n-\varepsilon$. Since

$-e \le \pi \le e$, $1-\varepsilon \le |\pi_j| \le 1$ for $j=1,\ldots,n$ and thus $d(\pi,\{-1,1\}^n) \le \varepsilon$. We conclude that

$d(P(\bar{A}), \{-1,1\}^n) \le \varepsilon$ and thus by Lemma 1, $P(\bar{A}) \cap \{-1,1\}^n \ne \emptyset$. $\square$

3. <u>The Complexity of Finding an Integer Element of a Polyhedron.</u>

It is well known (see for example Garey and Johnson [2]) that the

problem of determining whether there is an integer point in an H-cell is

NP-complete. In this section we show an analogous result for integral

containment in a W-cell. Consider

ICP2: <u>Given</u> $(\bar{U} \in Q^{n \times k})$

Question: Is there an integral n-vector $\pi \in X$, where

$$X = \{x \in R^n : x = \bar{U}\lambda, \ e^t\lambda = 1, \ \lambda \ge 0\}?$$

THEOREM 2. The problem ICP2 is NP-complete.

PROOF. Note first that ICP2$\in$NP since if $\pi \in X$ is integral, then we can

demonstrate that $\pi \in X$ by solving a linear program in polynomial time.

To show that ICP2 is NP-complete, we carry out a transformation from

the following 0-1 knapsack problem.

<u>Input</u>. Integers $a_1,\ldots, a_n$, b

<u>Question</u>. Is there a vector $y \in \{0,1\}^n$ such that $\sum_{i=1}^{n} a_i y_i = b$?

The above problem is known to be NP-complete.

Suppose that $a_1,\ldots, a_n$, b is an instance of the above knapsack

problem. We transform this instance into a problem in modular arithmetic as

follows: Are there vectors λ, s satisfying:

$(\sum_{j=1}^{n} a_j\lambda_j - b\lambda_{n+1})$ is integral, $\qquad\qquad$ (1a)

$(n+1)\lambda_j$ is integral for $j=1,\ldots, n+1$, $\qquad$ (1b)

$(2n)^{-1}(\lambda_j + s_j - \lambda_{n+1})$ is integral for $j=1,\ldots,n$, $\qquad$ (1c)

$s_1 + \ldots + s_n + \lambda_1 + \ldots + \lambda_{n+1} = 1$ $\qquad\qquad$ (1d)

$s, \lambda \ge 0$ $\qquad\qquad$ (1e)

First note that (1a) - (1e) is a special case of ICP2 in which U has 2n+1 columns each of which is in $\mathbb{R}^{2n+2}$.

We claim that there is a feasible solution to system (1) if and only if there is a solution to the knapsack problem.

Suppose first that $y \in \{0, 1\}^n$ is feasible for the knapsack problem. Let $\lambda_j = y_j/(n+1)$ for $j=1,\ldots,n$ and let $s_j = 1/(n+1) - \lambda_j$. Finally, let $\lambda_{n+1} = 1/(n+1)$. It is easy to verify that λ, s satisfy (1).

Suppose next that λ, s satisfy (1). If we subtract each of the n constraints of (1c) from $(2n)^{-1}$ of constraint (1d), we obtain the constraint

$$(n+1/2n)\lambda_{n+1} - 1/2n \text{ is integral} \qquad (1f).$$

Since $0 \le \lambda_{n+1} \le 1$, we conclude from (1f) that

$$\lambda_{n+1} = 1/(n+1) \qquad (1g).$$

We conclude from (1g), (1c) and (1d) that

$$\lambda_j + s_j = 1/(n+1) \quad \text{for} \quad j=1,\ldots, n \qquad (1h)$$

and by (1h) and (1b) we conclude that

$$\lambda_j = 0 \text{ or } 1/(n+1) \quad \text{for} \quad j=1,\ldots, n \qquad (1i).$$

From (1g), (1i) and (1a) we conclude that $y=(y_1,\ldots, y_n)$ is feasible for the knapsack problem, where $y_j=(n+1)\lambda_j$, $j=1,\ldots, n$, completing the proof. $\square$

Summary. In certain cases, the representation of a polyhedron as an H-cell or a W-cell drastically affects the computational complexity of the underlying problem. For the set containment problem, involving polyhedra and/or closed solid balls, the problem is solvable as a linear program and hence is polynomial, for the cases (H,H), (B,H), (W,H), (W,W), (W,B), and (B,B), see [1]. This note shows that the remaining three cases (H,B), (H,W), and (B,W) are NP-complete.

As regarding the determination of an integer point in a given polyhedron, this note has shown that the problem is NP-complete, irrespective of the representation of the polyhedron as an H-cell or a W-cell.

References

[1]   Eaves, B.C., and R.M. Freund, "Optimal Scaling of Balls and Polyhedra",
      Mathematical Programming 23 (1982), 138-147.

[2]   Garey, M.R. and D.S. Johnson, _Computers and Intractibility_, W.H. Freeman
      and Co., San Francisco, 1979.

[3]   Karp, R.M., "Reducibility among Combinatorial Problems", in R.E. Miller
      and J.W. Thatcher (eds.) _Complexity of Computer Computations_, Plenum Press,
      New York, 1972.

BASEMENT